Math 246C Lecture 12 Notes

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1 Green's Functions on Riemann Surfaces

1.1 Green's functions on Riemann surfaces

Let X be a Riemann surface. Take $x \in X$, let $z : U \to V$ be a complex chart, and let D be a parametric disc with $x \in D$ and $\overline{D} \subseteq U$ such that $z(x) = 0$. Let $\mathcal{F}$ be a family of continuous subharmonic functions $X \setminus \{x\} \to [-\infty, \infty)$ such that

1. For every $u \in \mathcal{F}$, there is a compact $K \subsetneq X$ such that $u|_{X \setminus K} = 0$.
2. For every $u \in \mathcal{F}$, $u(y) + \log|z(y)|$ is bounded above for y in a neighborhood of X.

$\mathcal{F}$ is a **Perron family** on $X \setminus \{x\}$.

Remark 1.1. The second condition does not depend on the choice of the parametric disc.

Set

$$G_x(y) = \sup_{u \in \mathcal{F}} u(y).$$

Definition 1.1. If $G_x < \infty$, then we say that the harmonic function G_x on $X \setminus \{x\}$ is a **Green's function** for X with pole at $x \in X$.

If $G_x \equiv \infty$, then we say that Green's function does not exist. To give an example where it does exist, first recall the Lindelöf maximal principle:

Theorem 1.1 (Lindelöf maximum principle[1]). *Let $\Omega \subseteq \mathbb{C}$ be open and bounded, and let $u \in SH(\Omega)$ be bounded above. If*

$$\limsup_{z \to \zeta} u(z) \leq M \qquad \forall \zeta \in \partial\Omega \setminus F,$$

where F is finite, then $u \leq M$ in all of Ω.

[1] This name is not completely standard but sometimes appears in the literature.

Example 1.1. Let $X = \{|z| < 1\}$. We claim that when $|a| < 1$, Green's function G_a exists, and

$$G_a(z) = \log \left| \frac{1 - \overline{a}z}{z - a} \right|.$$

Let $u \in \mathcal{F}$. Then

$$u(z) - G_a(z) = u(z) + \log \left| \frac{z - a}{1 - \overline{a}z} \right|,$$

which is subharmonic on $D \setminus \{a\}$, bounded above, and equals zero on ∂D. By the Lindelöf maximum principle, $u - G_a \leq 0$ on $D \setminus \{a\}$. We also notice that for every $\varepsilon > 0$, the function $\max(G_a(z) - \varepsilon, 0) \in \mathcal{F}$. The claim follows.

Example 1.2. If $X = \mathbb{C}$, then G_0 does not exist: consider $\max(\log R/|z|, 0)$ for large R.

Proposition 1.1. *Let $x \in X$, and let $z : D \to \mathbb{C}$ be a parametric disc with $z(x) = 0$. Assume that G_x exists. Then $G_x > 0$ on $X \setminus \{x\}$, and $G_x(y) + \log |z(y)|$ extends to a harmonic function on D.*

Proof. Let

$$u_0(y) = \begin{cases} \log \frac{1}{|z(y)|} & y \in D \setminus \{x\} \\ 0 & y \in X \setminus D. \end{cases}$$

Then $u_0 \in \mathcal{F}$. The function u_0 is subharmonic on $X \setminus \{x\}$, as $\max(\log(1/|z|), 0)$ is subharmonic on $\mathbb{C} \setminus \{0\}$. Then $u_0 \geq 0$, so $G_x \geq 0$ on $X \setminus \{x\}$, and $G_x > 0$ on D. By the maximum principle, $G_x > 0$ on $X \setminus \{x\}$.

Let $u \in \mathcal{F}$. Then $u(y) + \log |z(y)|$ is subharmonic in $D \setminus \{x\}$ and bounded above. By the Lindelöff maximum principle,

$$u(y) + \log |z(y)| \leq \sup_{\partial D} u \leq \sup_{\partial D} G_x < \infty, \qquad y \in D \setminus \{x\}.$$

So

$$G_x(y) + \log |z(y)| \leq \sup_{\partial D} G_x, \qquad y \in D \setminus \{x\}.$$

Also,

$$G_x(y) + \log |z(y)| \geq u_0(y) + \log |z(y)| = 0, \qquad y \in D \setminus \{x\}.$$

It follows that the bounded harmonic function $G_x(y) + \log |z(y)|$ extends harmonically to D (the singularity at x is removable). $\square$

Remark 1.2. It follows that $G_x(y) > 0$ is superharmonic on X. This explains why $\mathbb{C}$ does not admit any Green's functions; $-G_x$ would be a bounded subharmonic function on $\mathbb{C}$, but such a function does not exist.

1.2 Uniformization theorem, case 1

Theorem 1.2 (Uniformization, Case 1)**.** *Let X be a simply connected Riemann surface. The following conditions are equivalent:*

1. *$G_x(y)$ exists for some $x \in X$.*
2. *$G_x(y)$ exists for all $x \in X$.*
3. *There exists a holomorphic bijection $\varphi : X \to \{z : |z| < 1\}$.*

Proof. (3) $\implies$ (2): Let $\varphi : X \to \{|z| < 1\}$ be a holomorphic bijection, and let $x \in X$. We can assume that $\varphi(x) = 0$ (by composing φ with a Möbius transformation). Let $v \in \mathcal{F}_x$. Then $v(y) + \log|\varphi(y)|$ is subharmonic on $X \setminus \{x\}$, bounded above, and ≤ 0 far away from x. By the Lindelöf maximum principle, $v(y) + \log|\varphi(y)| \leq 0$ on $X \setminus \{x\}$. So $G_x = \sup_{v \in \mathcal{F}} v$ exists.

(2) $\implies$ (1): This is a special case.

(1) $\implies$ (3): Assume that G_x exists for some $x \in X$. By the proposition, $G_x(y) + \log|z(y)|$ is harmonic in the parametric disc $z : D \to \{|z| < 1\}$ (where $z(x) = 0$). Then there exists $f \in \mathrm{Hol}(D)$ such that $G_x(y) + \log|z(y)| = \mathrm{Re}(f(y))$ for $y \in D$. Let $\varphi(y) := z(y)e^{-f(y)}$. Then $\varphi(x) = 0$, φ is holomorphic, and $|\varphi(y)| = e^{-G_x(y)} < 1$ for all $y \in D$. We claim that φ continues holomorphically to all of X so that this holds globally on X. $\square$

We will prove the last part of this case next time.